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# Lie symmetries, nonlinear equations of motion and new Ermakov systems

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**Abstract.** It is shown that a Lagrangian of the form $L=\frac{1}{2}(\dot{\rho}^2-\omega^2(t)\rho^2)+\tilde{G}(t)\tilde{F}(k(t)\rho)$, said to be in factored form, yields an equation of motion that is equivalent to the most general equation derivable via Noether's theorem from the unfactored Lagrangian $L=\frac{1}{2}\dot{\rho}^2-\tilde{P}(\rho,t)$. In view of this equivalence, the theory of extended Lie groups is applied to the factored nonlinear equation of motion $\ddot{\rho}+\omega^2(t)\rho=G(t)F(k(t)\rho)$ to obtain its Lie symmetries. The latter are obtained when $G(t)$ and $k(t)$, initially arbitrary, are determined in terms of a function $x(t)$ which satisfies the auxiliary equation $\ddot{x}+\omega^2(t)x=K/x^3$. It is then possible with the auxiliary equation and the equation of motion to form a coupled pair of nonlinear equations, an Ermakov system, whose first integral is not invariant under the action of the symmetry group, in contrast to previous Ermakov systems.

## 1. Introduction

The existence of invariants (first integrals) for time-dependent nonlinear equations of motion can be useful in solving such equations. An excellent example is the invariant

$$I=\tfrac{1}{2}(x\dot{\rho}-\dot{x}\rho)^2+\int^{x/\rho}f(\lambda)\,\mathrm{d}\lambda+\int^{\rho/x}g(\lambda)\,\mathrm{d}\lambda \tag{1.1a}$$

for the coupled pair of equations

$$\ddot{\rho}+\omega^2(t)\rho=f(x/\rho)/(\rho^2x) \tag{1.1b}$$

and

$$\ddot{x}+\omega^2(t)x=g(\rho/x)/(x^2\rho) \tag{1.1c}$$

where overdots indicate differentiation with respect to $t$. The quantity $I$ is invariant, i.e. $\dot{I}=0$, if $\rho$ is any solution to (1.1$b$) and $x$ is any solution to (1.1$c$). The pair of equations (1.1$b$) and (1.1$c$) together with invariant (1.1$a$) is called an Ermakov system. A review of the literature on Ermakov systems with their applications is found in Ray (1981).

A natural question that arises in studying Ermakov systems, and the question we address in this paper, is the following. To what extent can one start from an equation of motion of a more arbitrary form, say,

$$\ddot{\rho}+\omega^2(t)\rho=P(\rho,t) \tag{1.2}$$

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and determine the explicit form of $P(\rho, t)$ such that (1.2) will possess an explicitly time-dependent invariant of the nature of (1.1*a*)? Several different approaches have been used to study this question. Probably the most successful approach so far is the application of Lutzky's version of Noether's theorem (Lutzky 1978a, Ray and Reid 1979a, Ray 1980). We briefly review Lutzky's approach below.

A system described by the Lagrangian $L(\rho, \dot\rho, t)$ has a Noether symmetry associated with the group operator

$$X = \xi(\rho, t)\partial/\partial t + \eta(\rho, t)\partial/\partial\rho \tag{1.3}$$

if there exists a function $\mathscr{F}(\rho, t)$ such that

$$X^{(1)}L + \dot\xi L = \dot{\mathscr{F}} \tag{1.4}$$

where $X^{(1)}$ is the generator of the first extended group, i.e.

$$X^{(1)} = X + (\dot\eta - \dot\xi\dot\rho)\partial/\partial\dot\rho. \tag{1.5}$$

If the system possesses a Noether symmetry then the Noether invariant follows from

$$I = \xi(\dot\rho\partial L/\partial\dot\rho - L) - \eta\partial L/\partial\dot\rho + \mathscr{F}. \tag{1.6}$$

Ray and Reid (1980) applied Noether's theorem to the Lagrangian

$$L = \tfrac{1}{2}(\dot\rho^2 - \omega^2(t)\rho^2) + \tilde{P}(\rho, t) \tag{1.7}$$

where $\tilde{P}$ is initially an arbitrary function. For this Lagrangian Noether's theorem leads to the Ermakov system

$$\ddot\rho + \omega^2(t)\rho = f(x/\rho)/(\rho^2 x) \tag{1.8a}$$

$$\ddot x + \omega^2(t)x = K/x^3 \qquad K = \text{constant} \tag{1.8b}$$

with Noether invariant

$$I = \tfrac{1}{2}(x\dot\rho - \dot x\rho)^2 + \tfrac{1}{2}K(\rho/x)^2 + \int^{x/\rho} f(\lambda)\,\mathrm{d}\lambda. \tag{1.8c}$$

This Ermakov system is, of course, a special case of (1.1). In system (1.8) the auxiliary function $x(t)$ is associated with the time part of the symmetry operator, i.e. $\xi = x^2(t)$. The fact that $\xi$ does not depend upon $\rho$ follows from Noether's theorem. In arriving at system (1.8) we assumed that function $\tilde{P}(\rho, t)$ was of the factored form

$$\tilde{P}(\rho, t) = \tilde{G}(t)\tilde{F}(k(t)\rho). \tag{1.9}$$

More recently, Leach (1981a) applied Noether's theorem to the Lagrangian

$$L = \tfrac{1}{2}\dot q^2 - V(q, t) \tag{1.10}$$

without the factorisation assumption (1.9). He is able to do so by solving the Noether equation (1.4) directly using the method of characteristics. Leach's results give the equation of motion

$$\ddot q = \ddot x q/x + (B\dot x - \dot B x)/x^3 - \tilde{G}'/x^3 \tag{1.11}$$

where $x(t)$ and $B(t)$ are arbitrary functions and $\tilde{G} = \tilde{G}(q/x + \int (B/x^3)\,\mathrm{d}t)$. The prime on $\tilde{G}$ in (1.11) indicates differentiation with respect to the argument of $\tilde{G}$. Leach showed that the Noether invariant has the form

$$I = \tfrac{1}{2}(x\dot{q} - \dot{x}q + B/x)^2 + \tilde{G}\left(q/x + \int (B/x^3)\,\mathrm{d}t\right). \tag{1.12}$$

Here again $x(t)$ is associated with the time part of the Noether symmetry, $\xi = x^2(t)$. The invariant given by (1.12) is a first integral of (1.11) for arbitrary $x(t)$, $B(t)$, and $\tilde{G}$. Thus Leach arrives at the most general equation of motion of the form

$$\ddot{q} = -\partial V(q, t)/\partial q \tag{1.13}$$

which has a Noether invariant, namely (1.11).

We now compare Leach's results to those of Ray and Reid (1980). First we carry out a transformation to a new dependent variable

$$\rho = q + x \int (B/x^3)\,\mathrm{d}t. \tag{1.14}$$

Equations (1.11) and (1.12) then take forms

$$\ddot{\rho} = \ddot{x}\rho/x - \tilde{G}'/x^3 \tag{1.15}$$

$$I = \tfrac{1}{2}(x\dot{\rho} - \dot{x}\rho)^2 + \tilde{G}(\rho/x) \tag{1.16}$$

respectively. Transformation (1.14) has the effect of eliminating the function $B(t)$ in (1.11) and (1.12), and at this stage $x(t)$ remains an arbitrary function. We can introduce another *arbitrary* function $\omega^2(t)$ such that $x$ satisfies the equation

$$\ddot{x} = -\omega^2(t)x + K/x^3 \tag{1.17}$$

without loss in generality of the function $x(t)$. Using (1.17) in (1.15), we have

$$\ddot{\rho} + \omega^2(t)\rho = K\rho/x^4 - \tilde{G}'/x^3. \tag{1.18}$$

Now by replacing the arbitrary function $\tilde{G}(\rho/x)$ by another arbitrary function $f(x/\rho)$ such that

$$\tilde{G}(\rho/x) = \tfrac{1}{2}K(\rho/x)^2 + \int^{x/\rho} f(\lambda)\,\mathrm{d}\lambda \tag{1.19}$$

$$\tilde{G}' = K\rho/x - f(x/\rho)x^2/\rho^2 \tag{1.20}$$

we find

$$\ddot{\rho} + \omega^2(t)\rho = f(x/\rho)/(\rho^2 x). \tag{1.21}$$

Leach's invariant (1.12) now has the form

$$I = \tfrac{1}{2}(x\dot{\rho} - \dot{x}\rho)^2 + \tfrac{1}{2}K(\rho/x)^2 + \int^{x/\rho} f(\lambda)\,\mathrm{d}\lambda. \tag{1.22}$$

The net result of our rearrangement of Leach's results are equations (1.17), (1.21) and (1.22), from which it is clear that his results are equivalent to the earlier results of Ray and Reid (1980).

One important lesson we learn from this discussion is that the factorisation assumption (1.9) gives the most general results from Noether's theorem. Note that transformations of Ermakov systems such as (1.14) are further discussed by Sarlet

(1981) and Sarlet and Ray (1981). Such transformed Ermakov systems arise in plasma physics problems (Lewis 1981).

It is well known that Noether's theorem is not the most general group theoretical structure for the investigation of symmetries of differential equations (Lutzky 1978b). The Lie theory, which directly investigates symmetries of differential equations, is a more general structure than Noether's theorem. In the next section we apply Lie theory to investigate the symmetries of (1.2). This analysis will automatically include any Noether symmetries, since every Noether symmetry is a Lie symmetry.

## 2. Lie symmetry

We are interested in imposing Lie symmetry on the equation

$$\Omega(\rho, \ddot{\rho}, t) \equiv \ddot{\rho} + \omega^2(t)\rho - P(\rho, t) = 0. \tag{2.1}$$

The generator of the symmetry transformation takes the form

$$X = \xi(\rho, t)\partial/\partial t + \eta(\rho, t)\partial/\partial\rho. \tag{2.2}$$

We shall also need the generators $X^{(1)}$ and $X^{(2)}$ of the first and second extended groups; these have the forms

$$X^{(1)} = X + \eta^{(1)}\partial/\partial\dot{\rho} \qquad \eta^{(1)} = \dot{\eta} - \dot{\xi}\dot{\rho} \tag{2.3}$$

$$X^{(2)} = X^{(1)} + \eta^{(2)}\partial/\partial\ddot{\rho} \qquad \eta^{(2)} = \ddot{\eta} - \ddot{\xi}\dot{\rho} - 2\dot{\xi}\ddot{\rho} \tag{2.4}$$

respectively. The equation of motion (2.1) is invariant under the symmetry generated by $X$ if

$$X^{(2)}\Omega(\rho, \ddot{\rho}, t) = 0 \tag{2.5}$$

whenever

$$\Omega(\rho, \ddot{\rho}, t) = 0. \tag{2.6}$$

There is a large amount of literature on the Lie method, and a recent article by Meinhardt (1981) contains earlier references. Of particular interest to this paper is the work by Leach (1981b): If (2.5) has nontrivial solutions for $\xi(\rho, t)$ and $\eta(\rho, t)$ then (2.6) possesses a Lie symmetry, and an invariant may be found from the equations

$$X^{(1)}I(\rho, \dot{\rho}, t) = 0 \tag{2.7}$$

$$\mathrm{d}I(\rho, \dot{\rho}, t)/\mathrm{d}t = 0. \tag{2.8}$$

In order to compare this with our earlier results using Noether's theorem (Ray and Reid 1980) we impose the factorisation assumption on $P(\rho, t)$ in (2.1). This means that we seek conditions under which the equation

$$\ddot{\rho} + \omega^2(t)\rho - G(t)F(k(t)\rho) = 0 \tag{2.9}$$

has a Lie symmetry. Although the assumed form for $P(\rho, t)$ will certainly reduce the generality of the equation (1.2), we are optimistic since the factorisation assumption gives the general solution in the case of Noether's theorem as we have shown in the preceding section.

## 3. Application of extended Lie symmetry operator

In this section we apply the second extended symmetry operator (2.4) to equation of motion (2.9). Having assumed $\xi=\xi(\rho,t)$ and $\eta=\eta(\rho,t)$, we find that

$$\eta^{(1)}=\eta_t+\dot\rho(\eta_\rho-\xi_t)-\dot\rho^2\xi_\rho \tag{3.1}$$

$$\eta^{(2)}=\eta_{tt}+\dot\rho(2\eta_{\rho t}-\xi_{tt})+\dot\rho^2(\eta_{\rho\rho}-2\xi_{\rho t})-\dot\rho^3\xi_{\rho\rho}+\ddot\rho(\eta_\rho-2\xi_t-3\dot\rho\xi_\rho) \tag{3.2}$$

where the subscripts $\rho$, $t$ indicate partial differentiation with respect to those variables and the overdot means total differentiation with respect to $t$. The operations implied in (2.5) lead to the condition

$$\begin{aligned}&\eta^{(2)}+\omega^2\eta-GF'k\eta+2\omega\dot\omega\rho\xi-(\dot GF+GF'\dot k)\xi=0\\&F'=\partial F/\partial(k\rho)\end{aligned} \tag{3.3}$$

and the requirement that (3.3) be satisfied on the manifold of solutions of $\Omega$ is met by substituting $\ddot\rho$ from (2.9) into $\eta^{(2)}$ in (3.2). When $\ddot\rho$ is thus eliminated, (3.3) becomes an identity in $\dot\rho$. Setting equal to zero the different powers in $\dot\rho$ then yields the set of partial differential equations:

$$\dot\rho^3\Rightarrow\xi_{\rho\rho}=0 \tag{3.4}$$

$$\dot\rho^2\Rightarrow\eta_{\rho\rho}-2\xi_{\rho t}=0 \tag{3.5}$$

$$\dot\rho\Rightarrow 2\eta_{\rho t}-\xi_{tt}-3GF\xi_\rho+3\omega^2\rho\xi_\rho=0 \tag{3.6}$$

$$\dot\rho^0\Rightarrow\eta_{tt}+\omega^2\eta+(\eta_\rho-2\xi_t)(GF-\omega^2\rho)+2\omega\dot\omega\xi\rho-GF'\eta-\xi(\dot GF+GF'\dot k\rho)=0. \tag{3.7}$$

Equations (3.4) and (3.5) imply

$$\xi(\rho,t)=a(t)+b(t)\rho \tag{3.8}$$

$$\eta(\rho,t)=\dot b(t)\rho^2+c(t)\rho+d(t). \tag{3.9}$$

With use of these last two equations one may bring (3.6) into the form

$$3(\ddot b+\omega^2 b)\rho+2\dot c-\ddot a-3GFb=0 \tag{3.10}$$

which is an identity in $\rho$. We assume at this point that the arbitrary function $F(k\rho)$ is neither constant nor linear in its argument. These restrictions are not severe inasmuch as (2.9) represents linear oscillators when $F$ is constant or is linear. It now follows from (3.10) that

$$b\equiv 0 \tag{3.11a}$$

$$2\dot c-\ddot a=0 \qquad 2c=\dot a+\alpha. \tag{3.11b}$$

We point out that symmetries of the linear oscillator with constant frequency were found by Anderson and Davison (1974), Wulfman and Wybourne (1976), and Lutzky (1978b), while those of the linear oscillator with time-dependent frequency have been obtained by Eliezer (1979) and Leach (1980). We shall proceed from this point on the assumption that $F$ has no constant or linear term. Note that (3.11*b*), though simple, is highly significant because it introduces a non-Noether symmetry through the constant parameter $\alpha$.

Taking account of (3.11) we can reduce expressions (3.8) and (3.9) to

$$\xi = a(t) \tag{3.12}$$

$$\eta = \tfrac{1}{2}(\dot{a}+\alpha)\rho + d(t) \tag{3.13}$$

respectively. Consequently, (3.7) becomes the equation

$$(\tfrac{1}{2}\dddot{a}+2\omega^2\dot{a}-2\omega\dot{\omega}a)\rho+\ddot{d}+\omega^2 d-dkGF' + [\tfrac{1}{2}(\alpha-3\dot{a})G-a\dot{G}]F-\tfrac{1}{2}(2a\dot{k}+\dot{a}k+\alpha k)G\rho F'=0 \tag{3.14}$$

implicitly an identity in $\rho$ through $F$, $F'$, and $\rho F'$. Due to our assumptions above on $F$, defining equations for $a$ and $d$ follow from (3.14), i.e.

$$\dddot{a}+4\omega^2(t)\dot{a}+4\omega\dot{\omega}a=0 \tag{3.15}$$

$$\ddot{d}+\omega^2(t)d=0. \tag{3.16}$$

The remnant of (3.14) is still an identity in $\rho$, and, in order to proceed with the calculation, we make assumptions on the relationship between the functions $F$, $F'$, and $\rho F'$. Should a linear relation hold between those functions, or any two of them, then all, or a pair, of the terms remaining in (3.14) would become additive. We shall treat this situation in the next section. Certainly, if those three functions are linearly independent it is possible to write the following equations:

$$F \Rightarrow a\dot{G}+\tfrac{3}{2}\dot{a}G-\alpha=0 \tag{3.17}$$

$$\rho F' \Rightarrow a\dot{k}+\tfrac{1}{2}(\dot{a}+\alpha)k=0 \tag{3.18}$$

$$F' \Rightarrow d(t)=0. \tag{3.19}$$

Solutions of (3.17) and (3.18) are

$$a^{3/2}G=\exp\left(\tfrac{1}{2}\alpha\int\frac{\mathrm{d}t}{a}\right) \tag{3.20}$$

$$a^{1/2}k=\exp\left(-\tfrac{1}{2}\alpha\int\frac{\mathrm{d}t}{a}\right) \tag{3.21}$$

respectively. Thus the previously arbitrary functions $G$ and $k$ are specified by (3.20) and (3.21), respectively, if the equation of motion (2.9) is to admit Lie symmetries. Upon multiplying (3.15) by $a$, integrating, and using the substitution

$$a=x^2 \tag{3.22}$$

it is possible to obtain an auxiliary equation identical to (1.8*b*).

We are now able to state the first important result of this section. The second extended Lie symmetry operator (2.4) applied to the equation of motion

$$\ddot{\rho}+\omega^2(t)\rho-G(t)F(k(t)\rho)=0 \tag{3.23}$$

has the effect of determining $G(t)$ and $k(t)$ in terms of $a(t)=x^2$ and $\alpha$ such that the equation

$$\ddot{\rho}+\omega^2(t)\rho=\exp(\tfrac{1}{2}\alpha\tau)F[\exp(-\tfrac{1}{2}\alpha\tau)\rho/x]/x^3 \qquad \mathrm{d}\tau=\mathrm{d}t/x^2 \tag{3.24a}$$

$$\ddot{x}+\omega^2(t)x=K/x^3 \tag{3.24b}$$

has the Lie point symmetry generator

$$X = x^2\partial/\partial t + (x\dot{x} + \tfrac{1}{2}\alpha)\rho\partial/\partial\rho. \tag{3.25}$$

Note that the pair of equations (3.24) is the Lie analogy to the Ermakov system (1.8). One may find an invariant for the system (3.25) in the manner of Ermakov (1880), i.e. by eliminating the frequency $\omega^2(t)$ between those two equations (Ray and Reid 1979b). The result is

$$I = \tfrac{1}{2}(x\dot{\rho} - \dot{x}\rho)^2 + \tfrac{1}{2}K(\rho/x)^2 - \int^t \exp(\tfrac{1}{2}\alpha\tau)F(r)(x\dot{\rho} - \dot{x}\rho)x^{-2}\,\mathrm{d}t' \tag{3.26}$$

for which $\mathrm{d}I/\mathrm{d}t = 0$ provided both equations in (3.24) are satisfied. However, this constant $I$ is not a Lie invariant because condition (2.7) above is not satisfied by $I$ in (3.26). We thus arrive at a second result, i.e. given the system of equations (3.24), the elimination of frequency technique leads to an invariant which is not associated with the symmetry group of the differential equation (3.24*a*). We shall call (3.26) an Ermakov invariant based solely on its derivation, and we merely point out its existence. No attempt is made here to investigate the significance or usefulness of (3.26), the integrand in which depends explicitly on solutions $\rho(t)$ and $x(t)$. For the case $\alpha = 0$, results (3.24*a*), (3.24*b*) and (3.26) are equivalent to our earlier results (Ray and Reid 1980) based on Noether's theorem. In that case, the Ermakov invariant is a Noether (and Lie) invariant of the symmetry group. We discuss in detail a generalisation of system (3.25) and (3.26) in another paper (Ray and Reid 1982).

We remark that the argument, $r$, of $F$ in (3.24ₐ) and (3.26) is identical to a change of variables

$$r = (\rho/x)\exp(-\tfrac{1}{2}\alpha\tau) \qquad \mathrm{d}\tau = \mathrm{d}t/x^2 \tag{3.27}$$

found and employed by Leach (1981b). (Actually, Leach's transformation contains an additive term that vanishes when $d$ is zero, as is the case here.) Simultaneously transforming both dependent and independent variables, (3.27) turns equation of motion (3.24*a*) into the autonomous equation

$$r'' + \alpha r' + Mr = F(r) \qquad r' = \mathrm{d}r/\mathrm{d}\tau \tag{3.28}$$

where $M = K + \frac{1}{4}\alpha^2$. In $r$, $\tau$ coordinates, invariant (3.26) takes the form

$$I = \tfrac{1}{2}(r' + \alpha r)^2\exp(\alpha\tau) + \tfrac{1}{2}K(\rho/x)^2\exp(\alpha\tau) - \int \exp(\alpha\tau)(r' + \alpha r)F(r)\,\mathrm{d}\tau \tag{3.29}$$

while the symmetry group generator (3.25) reduces to the simple form (Leach 1981b)

$$\tilde{X} = \partial/\partial\tau. \tag{3.30}$$

For further discussion of (3.29) refer to Ray and Reid (1982).

## 4. A special case

In this section we again direct our attention to identity (3.14) for the special case when $F$ and $\rho F'$ are linearly related, which occurs when

$$F(\rho) = \tilde{K}\rho^m \qquad \rho F' = mF \qquad \tilde{K} = \text{constant}. \tag{4.1}$$

We shall set the function $k(t)$ at unity, and we shall continue to exclude the constant and the linear form of $F$. Under the foregoing assumptions (3.14) becomes

$$[\tfrac{1}{2}\dddot{a}+2\omega^2\dot{a}+2\omega\dot{\omega}a]\rho+\ddot{d}+\omega^2 d-m\tilde{K}dG\rho^{m-1}$$
$$-[\tfrac{1}{2}(m+3)\dot{a}+a\dot{G}/G+\tfrac{1}{2}(m-1)\alpha]G\tilde{K}\rho^m=0. \qquad (4.2)$$

The special status of the exponent $m=2$ is apparent in (4.2). When $m\neq 2$, it follows that (3.15) still holds for $a$, that $d\equiv 0$ and that

$$a^{(m+3)/2}G=\exp\left(-\tfrac{1}{2}(m-1)\alpha\int\frac{\mathrm{d}t}{a}\right). \qquad (4.3)$$

This last equation is also valid for $m=2$. The system of equations

$$\ddot{\rho}+\omega^2(t)\rho=\tilde{K}\exp[-\tfrac{1}{2}(m-1)\alpha\tau](\rho/x)^m/x^3 \qquad (4.4a)$$

$$\ddot{x}+\omega^2(t)x=K/x^3 \qquad m\neq 2 \qquad (4.4b)$$

now results instead of (3.24). For this case, an invariant analogous to (3.26) takes the form

$$I=\tfrac{1}{2}(x\dot{\rho}-\dot{x}\rho)^2+\tfrac{1}{2}K(\rho/x)^2-\tilde{K}(m+1)^{-1}(\rho/x)^{m+1}\exp[-\tfrac{1}{2}(m-1)\alpha\tau]$$
$$+\alpha\tilde{K}(m-1)(m+1)^{-1}\int^t\exp[-\tfrac{1}{2}(m-1)\alpha\tau](\rho/x)^{m+1}x^{-2}\,\mathrm{d}t'. \qquad (4.5)$$

Consider next the case when $m=2$. The terms in (4.2) now combine in such a way to yield a more complicated equation for $a$, i.e.

$$\dddot{a}+4\omega^2\dot{a}+4\omega\dot{\omega}a-4\tilde{K}da^{-5/2}\exp\left(-\tfrac{1}{2}\alpha\int\frac{\mathrm{d}t}{a}\right)=0 \qquad (4.6)$$

where $a$ is coupled with the function $d$. In this case, $d$ satisfies (3.16). Equation (4.6) is identical to a result in Leach (1981b). It is clear from (3.10) that the function $b(t)$ will never be a consideration in the discussion of the symmetry groups of nonlinear ordinary second-order differential equations of the type (2.9). For the linear oscillator equation (see, e.g., Eliezer 1979) it is known that $b$ satisfies a second-order equation identical to (3.16). In view of the fact that any second-order ordinary differential equation can have at most an eight parameter group of point symmetries, it follows that the nonlinear second-order equation (2.9) may have at most a six parameter group. We note that the nonlinear equation at hand, i.e.

$$\ddot{\rho}+\omega^2(t)\rho=G(t)\rho^2 \qquad G(t)=\tilde{K}\exp(-\tfrac{1}{2}\alpha\tau)a^{-5/2} \qquad (4.7)$$

where $\mathrm{d}\tau=\mathrm{d}t/a$, where $a$ satisfies (4.6) and where $d$ satisfies (3.16), possesses the six parameter group of point symmetries.

## 5. Conclusion and discussion

Assuming a Lagrangian of the form

$$L=\dot{\rho}^2-\tilde{P}(\rho,t) \qquad (5.1)$$

Leach (1981a) found the most general equation of motion that has a Noether symmetry by solving directly the equation

$$X^{(1)}L+\dot{\xi}L=\dot{\mathscr{F}}. \tag{5.2}$$

We have shown in this paper that Leach's solution is equivalent to applying the Noether symmetry group operator to the Lagrangian

$$L=\tfrac{1}{2}(\dot{\rho}^2-\omega^2(t)\rho^2)+\tilde{G}(t)\tilde{F}(k(t)\rho) \tag{5.3}$$

in which the function $\tilde{P}(\rho, t)$ appears in the factored form shown in (5.3). Encouraged by this equivalence of results, we have applied the Lie theory of extended groups to the nonlinear equation of motion

$$\ddot{\rho}+\omega^2(t)\rho=G(t)F(k(t)\rho). \tag{5.4}$$

Using the Lie theory we have discovered a more general type of Ermakov system than follows from Noether's theorem. The system has the form

$$\ddot{\rho}+\omega^2(t)\rho=\exp(\tfrac{1}{2}\alpha\tau)F(\exp(-\tfrac{1}{2}\alpha\tau)\rho/x)/x^3 \tag{5.5a}$$

$$\ddot{x}+\omega^2(t)x=K/x^3 \tag{5.5b}$$

and

$$I=\tfrac{1}{2}(x\dot{\rho}-\dot{x}\rho)^2+\tfrac{1}{2}K(\rho/x)^2-\int\exp(\tfrac{1}{2}\alpha\tau)F(r)(x\dot{\rho}-\dot{x}\rho)x^{-2}\,\mathrm{d}t \tag{5.5c}$$

where $r=(\rho/x)\exp(-\frac{1}{2}\alpha\tau)$ and $\mathrm{d}\tau=\mathrm{d}t/x^2$. This system goes over into our earlier results obtained using Noether's theorem for $\alpha=0$. A distinctive feature of the new Ermakov system (5.5) is the fact that the invariant is not associated with the symmetry group of the differential equation for $\alpha\neq 0$. This fact is somewhat surprising since the form of the equation of motion (5.5*a*) and of the auxiliary equation (5.5*b*) follow by imposing Lie symmetry, while the invariant (5.5*c*) is obtained by combining these two equations in a simple manner. As far as we are aware, (5.5) is the first Ermakov system which has the property that its first integral is not invariant under the action of its symmetry group. A generalisation of system (5.5) and the role of nonlinear superposition will be treated elsewhere.

As a final remark, we note that the nonlinear equation (5.5*a*) is unusually well endowed with symmetries, having a total of four symmetry parameters. Three of them, forming a subgroup (Eliezer 1979, Leach 1980, 1981b, Lutzky 1978b), are attributed to the function $a(t)$, which satisfies the third-order equation (3.15), while the fourth is associated with the non-Noether parameter $\alpha$. For the special case $F(\rho)=\tilde{K}\rho^2$, the equation of motion for the quadratically anharmonic oscillator has six symmetries. In this case, the two additional symmetries accrue to the function $d(t)$.

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*Note added in proof.* When those functions $a$, $b$, $c$, $d$ that appear in group operator (2.2) via (3.8), (3.9), (3.11) also appear in the differential equation whose symmetry is sought then the symmetry group for *that* equation is one dimensional. In contrast, when these functions do not appear in the equation under study then the arbitrary constants of integration $\alpha_i$ in functions $a$, $b$, $c$, $d$ lead to several independent group operators $X_i$ which yield a multi-parameter Lie symmetry group. For example (3.24$a$), in which parameter $\alpha$ and the function $x^2 = a$ appear, has only a one-parameter symmetry group corresponding to generator (3.25). On the other hand, if the right side of (3.24$a$) is simply $1/\rho^3$ that equation then has a three-parameter Lie symmetry group with $\alpha = 0$ in generator (3.25). Note that the multi-parameter groups for coupled systems discussed in this paper are therefore not Lie symmetry groups in the conventional sense. These groups are generated by allowing parameters in a differential equation to vary as the *same* parameters in the group operator vary. Our multi-parameter group is therefore associated with a set of differential equations. Note that none of the calculations in this paper are affected by the foregoing remarks.

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